Some Cognitive Obstacles Faced By ‘A’ Level Mathematics Students in Understanding Inequalities: A Case Study of Bindura Urban High Schools

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Abstract

In this study Mathematics lecturers from Bindura University of Science Education worked with mathematics teachers to determine the cognitive obstacles faced by ‘A’ level mathematics students in Bindura Urban High Schools. Knowledge of cognitive obstacles faced by ‘A’ level mathematics students is a key component of mathematics teachers’ knowledge for teaching mathematics referred to as pedagogical content knowledge (Shulman, 1986). The hope was that knowledge of students’ cognitive obstacles would enable teachers to provide high quality instruction for all students.

The lesson study (a teaching improvement and knowledge building process) was considered as the appropriate theoretical framework to use to capture the ‘A’ level students’ cognitive obstacle when solving problems involving inequalities.

The results of the study seem to indicate the major cognitive obstacles of the ‘A’ level mathematics students in coming to understand inequalities are overreliance on deceptive intuitive mathematical knowledge, undue focus on mathematical procedures, compartmentalization of mathematics topics and the representations of mathematical concepts.

Keywords: cognitive obstacles, understanding, inequalities, mathematics students

Background to the study

Mathematics maintains an enviable position and shall continue to remain so in our everyday life. No wonder it remains as a core subject on the curricular from kindergarten to the University. However, poor students’ performance in science subjects in Secondary Schools such as Mathematics and Sciences is an issue that has been well known and discussed by many...
people for so long in Zimbabwe (Mtetwa, 2000). Mathematics has the highest failure rates in all public examinations. Math phobia or the fear and hatred of mathematics prevalent in our schools is not only restricted to the low achievers but unexpectedly cuts across the rank and file of all students even including the seemingly brighter ones. It is a concern since good students’ performance in science subjects is essential for social and economic development. The Government of Zimbabwe was cognizant of the fact that students’ poor performance in mathematics was partly due to shortage of Mathematics teachers, hence, in 1995, the Government made a decision to address the problem of the shortage of science teachers locally by setting up Bindura University College of Science Education whose thrust was to produce Mathematics and Science teachers. The hope was that this would result in qualitative improvement of learners’ scientific and technological literacy.

However, despite these ambitious strides there are massive failures in mathematics and science subjects in secondary schools. The most disturbing thing for Bindura University of Science Education is that the students’ performance in Mathematics and Science subjects was also low in Bindura Urban High schools which are supposed to benefit from their proximity to the university. As a proactive measure the Faculty of Science Education established a Sub-committee whose mandate was to find ways of improving the quality of learning in the schools. The terms of reference of the sub-committee were heavily informed by the concerns of the Ministry of Education, Sports, Arts and Culture. One of the major concern was poor students’ performance in Mathematics and Science subjects at both ‘O’ and ‘A’ Level in Bindura urban schools. Consequently, one of the terms of reference of the sub-committee was to organize workshops, clinics and refresher courses for both teachers and students in Mathematics, Chemistry, Biology and Geography.

Having been given the mandate to go into the schools and work with both learners and teachers created an opportunity for lecturers in the Mathematics Department of Bindura University of Science Education to work with teachers in determining the cognitive obstacles faced by ‘A’ level mathematics students in Bindura Urban High Schools. Knowledge of cognitive obstacles faced by ‘A’ level mathematics students is a key component of mathematics teachers’ knowledge for teaching mathematics referred to as pedagogical content knowledge (Shulman, 1986). This paper reports on cognitive obstacles faced by ‘A’ level mathematics students in understanding Inequalities. Knowledge of students’ cognitive obstacles would enable teachers to provide high quality instruction for all students.

**Theoretical framework of the study**

The Japanese lesson study process was considered as the appropriate theoretical framework to use to capture the ‘A’ level students’ cognitive obstacle when solving problems involving inequalities. Lesson study is a teaching improvement and knowledge building process that has origins in Japanese elementary education. Lesson study is a literal translation for the Japanese word Jugyokenkyu—jugyo means lesson and kenkyu means study or research. In Japanese lesson study teachers work in small teams to plan, teach, observe, analyze, and refine individual class lessons as shown in Figure 1.

Fernandez, Cannon, and Chokshi (2003) describe the development of three lenses for examining lessons in a lesson study process:
1. Teacher lens - to see how to sequence and connect learning experiences;
2. Student lens - to understand student thinking, anticipate student behavior, and learn to build student understanding; and,
3. Researcher lens - to see how to use the classroom as a laboratory for generating data-driven conclusions about pedagogy.

Teacher lens

A common misconception about lesson study is that the study is intended to determine the lesson’s effectiveness (e.g., whether students learn what they are supposed to learn and achieve the lesson’s goals). Of course this is an important question, and one that most teachers want to answer. However, the primary focus of lesson study is not what students learn, but rather how students learn from the lesson. It is a process by which teachers systematically examine their practice and their students’ learning to become more effective instructors. The process create an opportunity for teachers to create a deep and grounded reflection about the complex activities of teaching that can then be shared and discussed with other members of the profession.

At the lesson planning level the teachers identify a broad overarching goal or a vision of the type of student the educational community wishes to produce such as active problem-solvers. (Lewis and Tsuchida, 1998). They then identify and plan collaboratively a specific lesson topic that might address this goal while at the same time considering student backgrounds, and other initiatives designed to address the broad goal. One member of the group proceed to teach the lesson as designed while other group members and outsiders observe the class taking note of detailed notes regarding the reactions and engagement of both the teacher and the students. The lesson can be video-taped. The teacher and other group members gather soon after the lesson has been taught to share thoughts and insights and evaluate the success of the lesson in meeting its objectives. They discuss practical suggestions to improve the lesson. Based on experience and evidence, the lesson is often revised and taught again, and the process is repeated.

Researcher lens

When approaching the lesson study process from the point of view of a researcher the team develops a plan to investigate a research issue or a topic such as students’ cognitive obstacles in understanding inequalities. The plan specifies the type of data the team will collect, how observers will observe and record data during the lesson, the nature of the exercises in which students explain key ideas, both orally and in writing and how the team can put forward its work in a form that can be peer reviewed and built upon by others.
As colleagues observe the lesson they note what the teacher and the students say and do. The team focuses on student thinking, how learners make sense of the material, kinds of cognitive obstacles students have, how students answer questions, how their thinking changed during the lesson.

In the post lesson conference stage the teacher and other colleagues share thoughts and insights but as teacher-researchers they also make practical suggestions to improve the research, reflect on which insights gained might apply to future classroom settings. Besides reflecting on possible future research issues to be explored, they discuss possible changes to the study and where possible generalize research findings to other applicable contexts.

Student’s lens

The lesson study group practice what Yoshida (1999) refer to as cognitive empathy so as to make student thinking visible. Teachers put themselves in the position of a student imagining what it would be like to experience the material and lesson activities as a novice. In planning a lesson, these teachers besides considering what each solution says about student understanding and processing, they predict how students are likely to respond to specific questions, problems and exercises. Also in order to investigate student learning during the class period, teachers try to design a lesson that makes students’ thinking visible—that is, open to observation and analysis.

From the point of view of the teacher-researchers the lesson study process would culminate in a research study report. The research study report would include: the research issue under investigation, a description, discussion and summary of findings, conclusions about the lesson, especially with respect to student learning goals but also about the methods used to study it; and supplementary material such as data video clip of the lesson, transcribed think aloud protocols of students so that interested researchers could replicate the study.

Research question

The Japanese lesson study philosophy outlined above was used as a theoretical framework in answering the following research question.

- What are the ‘A’ level mathematics students’ cognitive obstacles when solving problems involving inequalities?

Literature review

The notion of a cognitive obstacle was first introduced in the realms of science by Bachelard (1938) and highlighted in mathematical education by Brousseau (198?). In their terms an obstacle is a piece of knowledge of the student that has in general been satisfactory for a time for solving certain problems, and so becomes anchored in the mind, but subsequently, when faced with new problems, it proves to be inadequate and difficult to adapt. Brousseau goes on to classify cognitive obstacles as ontogenetic, didactical, and epistemological. Herscovics (1989) used the term cognitive obstacle to refer to either the existing mental structure (or its
attempted use) or the structure of the new material. He believed that learning difficulties were of two basic types:

1. The learner attempts to map new material onto an existing mental structure which is valid in another domain but inappropriate for the knowledge to be learned.

2. The inherent structure of the new material might be such that the learner has no existing mental structure which would allow assimilation of the new material.

Kinds of cognitive obstacles stated by Herscovics identified from the work of Bachelard include: the tendency to rely on deceptive intuitive experiences, the tendency to generalize and obstacles caused by natural language.

On the other hand Cornu (1991) differentiates between four types of obstacles: cognitive obstacles, genetic and psychological obstacles, didactical obstacles, and epistemological obstacles. According to Cornu, cognitive obstacles are a product of the student’s previous experience and their internal processing of these experiences and are manifested when students encounter difficulties in the learning process. Genetic and psychological obstacles also referred to as ontogenic obstacles occur as a result of personal development of the student. Didactical obstacles are those that arise as a result of instructional choices and therefore, are avoidable through the development of alternative instructional approaches. Epistemological obstacles, in contrast, are those that arise regardless of the instructional approach, for their origin is in the nature of the mathematical concepts themselves. These descriptions by Cornu seem to give an impression that there is a clear distinction between these kinds of obstacles. However, Tall (1989) used the term ‘cognitive obstacle’ for epistemological obstacle. Like Herscovics, Tall’s view is that Bachelard defined the notion of epistemological obstacle in the context of the development of the scientific thinking in general and not in terms of individual learning experiences. Tall (1989) chooses to interpret the notion of cognitive obstacle in terms of Piagetian theory, where the learner, confronted with new ideas that cannot be fitted into his existing cognition, is unable to cope adequately with the new information. Sierpinska (1985, 5) argued that there is a “property of duality of epistemological obstacles”, that is, epistemological obstacles can manifest themselves in terms of coupled ways of knowing, which are incompatible while on the other hand obstacles can be characterized in terms of coexisting conceptual understandings and perspectives of mathematical knowledge. From this viewpoint, it is important when exploring epistemological obstacles to identify potentially incompatible ways of knowing, and the ways in which students’ conceptual understandings relate to particular perspectives of mathematical knowledge. The idea that ways of knowing function within a system of coexisting conceptual understandings and perspectives of mathematical knowledge aligns well with Harel’s (1998) Dual Assertion: the idea that not only do students’ ways of thinking affect the meanings students attribute to mathematical concepts (ways of understanding), but also that students’ ways of understanding affect their ways of thinking. Thus Sierpinski’s approach can be viewed as supporting the exploration of epistemological obstacles through the consideration of students’ ways of thinking and ways of understanding.

The preceding discussion has shown that differentiating the term cognitive obstacle from the other types of obstacles is a very complicated matter. Some authors believe that there is an overlap between cognitive obstacles and other types of obstacles. Others believe that there is a clear distinction between obstacles. A clear distinction between these obstacles in reality may not necessarily be simple because of the complex nature of knowledge acquisition. The
variations in how the obstacles are classified can be attributed to the fact that knowledge acquisition takes place in a very complex system of interaction. One such subsystem that could be mentioned consists of the teacher, the student, and the knowledge system (Brousseau, 1997). When a learner experiences an obstacle in learning, how are we to apportion the blame on the system of interaction? Is it not possible for a learner to experience an obstacle in the process of learning due to the nature of teaching and the teacher, or because of the nature of the subject matter or because of the learner’s genetic and personal development or a combination of the above? However, besides this complexity, there is evidence that students do encounter obstacles in acquiring knowledge. In this study the term ‘cognitive obstacle’ will be taken to mean any causes of stagnation or inertia in knowledge acquisition in the ‘A’ level mathematics classrooms.

Research Methodology

Target Group

In the study there were twelve and eight ‘A’ level students from two urban high schools respectively which are within five kilometres from Bindura University. The students were taught in one afternoon as one group at one of the high schools.

Data collection procedure

The researchers and the ‘A’ Level mathematics teachers collaboratively designed lesson plans for teaching inequalities which is a key component of the ‘A’ level mathematics syllabus. While one member was teaching the other members were writing detailed observations of students’ activities and written work during the lesson. The researchers did not rely on students’ written work only since we were aware that not all the students’ thinking is put down on paper but lots of information stay inside the brains. Hence we decided to use think-out-loud protocols as research instruments. Instead of observing how the teacher teaches, as in typical classroom observations, observers focused on how students responded to the lesson, which was designed by the team rather than by the person who happened to be teaching. The collective ownership of the lesson paved the way for public knowledge building. We gathered rich evidence related to the research study objectives, capturing the complexity of actual teaching and learning. The lessons were videotaped for future reference and review.

Data Analysis techniques

Documentary analysis of the students written work and post lesson reflective interviews played a pivotal role in data analysis. Soon after the lesson was taught we held a debriefing meeting to examine evidence related to ‘A’ level mathematics students’ cognitive obstacles and to reflect on the experience. We shared our observations and examined additional evidence from the lesson, such as student written work and the transcribed think-out-loud protocols, searching for patterns that could reveal important insights into cognitive obstacles.
Results and Discussion

Students’ cognitive obstacles manifested themselves when students were responding to the three parts task in which they were supposed to use their knowledge of the modulus function, transformation of functions, and solution of equations to solve an inequality task as shown below. The task was extracted from a past ‘A’ Level paper in which candidates were expected to score a maximum of 120 marks in 180 minutes implying that the task was to be answered within nine minutes. \[ y = |x - 1| - 2 \text{ and } y = -|x - 1| \]

(a) On the same axes, sketch the graphs of \( y = |x - 1| - 2 \) and \( y = -|x - 1| \)  
(b) Solve the equation \( |x - 1| - 2 = -|x - 1| \)
(c) Hence solve the inequality \( |x - 1| - 2 > -|x - 1| \)

Overreliance on deceptive intuitive mathematical knowledge, focusing on mathematical procedures and processes, compartmentalization of the mathematical ideas and mathematical representations proved to be the major causes of stagnation or inertia in knowledge acquisition in the ‘A’ level mathematics classrooms.

Overreliance on deceptive intuitive mathematical knowledge

Overreliance on intuitive mathematical knowledge proved to be a cognitive obstacle to the ‘A’ level mathematics students as they solved problems involving inequalities. Although some of the students knew that they were supposed to sketch the two functions by transforming the graph of the standard modulus function some were not able to do the transformations as shown in Figure 2

The graphs in figure 2 shows that the student instead of translating the standard modulus function one unit to the right, he translated it one unit to the left thereby obtaining the graph of \( y = |x + 1| \) which he labeled as the graph of \( y = |x - 1| \)
The transcript below illustrates the nature of the student’s cognitive obstacle when intuitive knowledge, which, according to Fischbein (1987) is a type of immediate, implicit, self-evident cognition that leads in a coercive manner to generalizations prior to any need for explicit justification or interpretation, is brought to the fore in solving problems.

Teacher: Just explain to me how you got the graph of \( y = |x - 1| \)

Pupil: I used the graph of \( y = |x| \)

Teacher: How did you use the ..... 

The student did not wait to hear the full question

Pupil: I just shifted the graph of \( y = |x| \) one unit to the left

Teacher: Should it be to the left or the right?

Pupil: Definitely it should be to the left

Teacher: What is your basis for translating to the left?

pupil: ...........

Teacher: can you check the accuracy of your graph by verifying for a few points say when \( x = -1, 0 \) and 1

The learner evaluated the corresponding \( y \)-coordinate when \( x = -1, 0 \) and 1 respectively and plotted the points as shown on the graph. After rechecking his calculations he said:

These points do not lie on the curve and yet they should lie on the curve.... there is no consistency between the graph and the coordinates I calculated.

Teacher: Can both your assertion that you should shift to the left and the set of coordinates which you calculated be correct at the same time?

Pupil: No... but I always have this feeling that if a number is negative such as this -1 then shift to the left. I would shift to the right if it was positive

Teacher: Now for argument’s sake can you shift the graph of \( y = |x| \) one unit to the right and note if there is consistency between the set of your coordinates and the resultant graph?

What was of great concern was that although the students had been taught transformations of functions and were preparing to write their ‘A’ level examinations in a month’s time the deceptive intuitions or presuppositions remained influential even after instruction. Although intuitions are not the perfect reliable sources of absolutely credible form of knowledge it seems one cannot eliminate intuitions i.e. the immediate, implicit, self-evident cognitions that lead in a coercive manner to generalizations. Kant (1980) remarked that intuition have to appear to be the primary source of absolute knowledge because that is their role: to create the appearance of certitude so as to act as safe landmarks in the course of reasoning activity. The transcript above exemplifies what Kant meant by intuitive knowledge acting as apparently safe landmarks in a reasoning endeavor.

For example, when the student relied on his intuition to shift the graph of \( y = |x| \) one unit to the left hoping to get the graph of \( y = |x - 1| \), mentally, his behavior can be described as being passive i.e. that action was not based on conscious and explicit logical reasoning, justification or sufficient empirical evidence. However the student became mentally active when he realized that his assertion was not compatible with his algorithmic knowledge (i.e. the set of
coordinates which he calculated and plotted). Although the student remarked that ‘there is no consistency between the graph and the coordinates I calculated’, what he fell short of saying was that there was a disequilibrium in his mind. Using the language of Ausebel (1988) the student was experiencing a cognitive dissonance. Hence the student became mentally active as soon as he started to act on his intuitive knowledge and algorithmic knowledge bases-rechecking his calculations, this time translating the curve one unit the right and noting any other disparities thereby hopefully resolving the dissonance. The very act of resolving the perceived cognitive dissonance is considered as a crucial stage in the process of learning mathematics (Ausebel, 1988).

Mathematics teachers should become more skeptical of the validity of intuitions since they are deceptive in so many instances and some propositions that would have been accounted true by intuition are repeatedly proved false by logic. Teachers should not accept any mathematical proposition based on intuitive convictions. However they should exploit the learners’ intuitive knowledge which they bring to the learning of mathematics as opportunities for creating cognitive dissonance in the learners’ mind. As soon as learners are helped to resolve these dissonances meaningful learning would have been attained.

**Focusing on mathematical procedures and processes**

Focusing on procedures and processes proved to be one major cognitive obstacle of the ‘A’ level mathematics students when solving problems on inequalities. The process-object duality nature of mathematical concepts was the source of ‘A’ level mathematics students’ cognitive obstacle. In the task the symbols such as \( y = |x - 1| - 2 \) has dual connotations as a process (a dynamic transformation of independent \( x \) values to get the dependent \( y \) values) and as an object (a function on which it is possible to perform actions on it such as translating or reflecting the function). Gray and Tall (1994) introduced the term ‘proceptual divide’ to explain the differences in thinking between learners who cling to remembered procedures and those who become increasingly flexible through the use of proceptual thinking. Proceptual thinking includes the use of procedures where appropriate and thinking about mathematical symbols as manipulable objects where appropriate. There was a marked difference in terms of how students who had an object conception of functions approached the task as opposed to students with a process or procedural conception of functions. While students possessing a strong proceptual understanding of functions in the task were able to come up with what Naidoo and Huntley (2011) termed proceptual solutions i.e. solutions realized with the integration of a particular process and a particular object, the majority of the students’ solution strategies ranged from totally wrong solutions to very long tedious solutions in which case the time and effort devoted were not commensurate with the maximum scores allocated. What was conspicuous about these solutions was that these students were focusing on procedures and processes. In this study we refer to these students as procedural thinkers.

Students’ responses to Task 1 shown in Figure 3 is an example of a solution given by student Y, which we classified as a proceptual solution after having listened to the thinking behind the answer.
The transcript illuminates the thinking behind the solution given in Figure 3.

Teacher: Can you explain to the class how you finally got the sketch of the function?

Student Y: We all know the graph of the standard modulus function \( f(x) = |x| \) passes through the origin and the obvious points (-1, 1) and (1, 1). The next step is to shift this standard modulus function one unit to the right to obtain the graph of the function \( f(x) = |x - 1| \). When this is translated two units downwards we get the graph of \( y = |x - 1| - 2 \). If I reflect the graph of \( f(x) = |x - 1| \) on the x-axis I get the graph of \( f(x) = -|x - 1| \). I also keep track of the movement of the obvious points of the standard modulus function for every transformation performed.

Teacher: I can see the answers to the (b) part of the question. Where are they coming from?

Student Y: The values of \( x \) for which the two functions are equal are the \( x \) coordinates at the points of intersection of the two functions. The two vertical lines show that the two functions are equal when \( x = 0 \) or 2. To get the solutions to the inequality we have to look for the range of values of \( x \) for which the graph of \( f(x) = |x - 1| - 2 \) is above the graph of \( f(x) = -|x - 1| \). The two horizontal arrows show that the inequality is valid when \( x < 0 \) or when \( x > 2 \).

The transcript above shows that this particular student got the solution effortlessly by switching between viewing the functions in the task as processes (calculating the corresponding values of \( y \) for some values of \( x \)) and seeing them in the condensed form as objects which can be manipulated to produce other functions.

However procedural thinking was a drawback for the procedural thinkers. Different situations activated different schemes in the student’s minds and led them to behave in an inconsistent manner responding to the task. For example we noted situations where students attempted to find either the derivatives or the intercepts of the functions. Obviously this approach was motivated by the students’ knowledge of the procedures they would use in sketching functions.
differentiable at all points having stationary points. There were also cases where some students attempted to sketch the function by constructing tables of values thereby showing lack of understanding the general characteristics of modulus functions. We noted that these students zeroed on these approaches intuitively at the mere sight of the phrase ‘sketch the graphs of’. When asked to justify the choice of their procedural approaches the students responded by saying that the phrase ‘sketch the graphs of’ demands that they embark on the chosen approaches.

Although the above approaches were not likely to yield correct solutions to the first part of the task in some cases the adopted solution procedures had the potential of producing correct answers provided they were error free. The two solutions of the equation and the inequality respectively below produced by the same student demonstrate how procedural thinking can impede students’ success in ‘A’ Level mathematics examinations.

In light of the process, object and procept conceptions characterization of student levels of understanding, we note that students having a process view of inequalities found the task very challenging. Consequently teachers should endeavor to embed skills that enable students to become flexible when solving mathematical problems. It entails that students must be taught to view mathematical concepts both as a process and as an object.

In Zimbabwe, the general perception is that the ‘O’ level secondary school teaching of mathematics tends to be fairly procedural and hence students who eventually proceed to study ‘A’ level mathematics are better equipped to deal with procedural problems than with conceptual problems. Teaching for procedural knowledge means teaching definitions, symbols and isolated skills in an expositorily manner without first focusing on building deep, connected meaning to support those concepts (Skemp, 1987). Teaching for conceptual knowledge at ‘A’ level, on the other hand, begins with posing problems that require students to reason flexibly. Throughout the solution process, students are expected to make connections to what they already know, thus allowing them to extend their prior knowledge and transfer it to new situations (National Council of Teachers of Mathematics, 2000). In order to solve the task the students need not only to draw from their knowledge of transformations, the modulus function and solutions of equations and inequalities but also need to have developed a proceptual conception of the mathematical concepts in the task.
Compartmentalization

Another cognitive obstacle which resulted in ‘A’ level mathematics students’ inability to solve problems involving inequalities was compartmentalization. Besides compartmentalizing the mathematical ideas the students were also compartmentalizing the representations of inequalities.

Compartmentalization of the mathematical ideas

In order to solve problems involving inequalities students have to integrate and bring to the fore their full understanding of the graphical and algebraic methods of solving equations and the various representations of functions especially modulus functions. However students tended to come up with different approaches to each of the three part problem. They came up with two or three unrelated procedures for each of the three parts of the task without exploiting the interrelatedness of the three part task.

After sketching the two functions they would proceed to solve the equation algebraically, forgetting that the roots of the equation could be read at the points of intersection of the functions sketched in the first part of the question. In other cases we noticed that after getting the roots of the equation the students would start afresh to solve the inequality by initially squaring both sides of the inequality. The students did not take cognizance of the fact that the answer to the inequality could be obtained by integrating the results of the first two parts of the question.

These results are consistent with Gray and Tall’s (1994) claim that a major source of the generative power of mathematics is in the use of symbols which are used ambiguously to evoke both a process of calculation and the product of that calculation. They related the divergence of success between the high achievers and the low achievers to the development of the procept conception by suggesting that the interpretations of mathematical concepts and their related symbolic representations as processes or objects leads to a proceptual divide between the less successful and the more successful. In their work with students they found out that the more able students were able to treat mathematical symbolism flexibly as processes or as objects, whichever was more appropriate in a given context while the less able tended to conceive mathematics as separate procedures to be done.

Compartmentalization of representations of inequalities

Inequalities can be represented algebraically, verbally and graphically. However, in this study, there was a tendency by the procedural learners to compartmentalize the algebraic, verbal and graphical representations of inequalities. Students’ inability to switch flexibly from one representational form to another seemed to be one of the major cognitive obstacles to a firm understanding of inequalities.

Compartmentalization of representations of inequalities was evident initially when procedural learner could not read the solution of the inequality from the graph. After having obtained the correct graphical representations of the two functions reading the solution of the inequality
was an uphill task for the procedural learners. The procedural learners were only able to come up with a verbal description of the solution i.e. $x$ is less than 0 or $x$ is greater than 2 after rewording the last part of the task to ‘For what values of $x$ is the graph of $y = |x-1|-2$ above the graph of $y = -|x-1|$. Obviously the reworded version of the question acted as a crutch for the procedural learners without which they would not have come up with the verbal description of the solution of the inequality. Poor understanding of the symbolic representation was evident again when students were asked to represent verbal description of the solution set using mathematical symbols. Some of the answers given by the students were:

Student1 0<x<2  
Student2 0<x>2  
Student3 0<x>2  
Student4 x<0>2

The illustration demonstrates that what the procedural students thought and could verbalize is correct but they were not able to write down correctly their thoughts using mathematical symbols and yet they were not astonished by the inherent contradiction. Mathematical symbolism was a major source of both success and distress in understanding inequalities. The conceptual thinkers were doing qualitatively different mathematics from the procedural thinkers. It is clear that the mathematics of the conceptual thinkers is conceived to be, for them, relatively simple since they have been able to de-compartmentalize the representations of inequalities, whilst the procedural thinkers are doing a different kind of mathematics which is hard due to compartmentalization of representations of inequalities. The inequalities are best understood if their mental representations are part of a network of representations.

**Conclusion**

Learners come to the learning of inequalities with different forms of mathematical knowledge including intuitive mathematical knowledge and mathematics teachers are being encouraged to base their teaching on this prior knowledge. However the findings of the current study show that intuitive knowledge can be insufficient, inappropriate, inaccurate or deceptive and that knowledge get in the way of new understanding. The secret to learning then is to be willing to unlearn the deceptive intuitive understandings, even if the intuitive understandings previously brought success in other topics. Success in mathematics depends partly on how fast one can unlearn because one can't always learn something new until one first let go of something else.

Pateman (2009) noted that one of the barriers that often stops unlearning from occurring is that our current school system is influenced by learning theorists who talk about acquisition and retention of knowledge, but they never refer to giving away or expulsion. The challenge is to try to turn this around so that students have the opportunity not just to learn new things, but also to unlearn deceptive intuitive knowledge and then relearn. It is the task of the skilled mathematics teacher to deconstruct, confront and challenge tacit knowledge so that unlearning and then relearning can be facilitated.

The results of the study also show that learners had difficulties in translating from one mode of representation of inequalities to another and this particular phenomenon reveals a cognitive difficulty that arises from the need to accomplish flexible and competent translation back and
forth between different kinds of mathematical representations (Duval, 2002). Since a concept such as an inequality is not acquired when some components of mathematical thought are compartmentalized, teaching needs to accomplish the breach of compartmentalization, i.e., de-compartmentalization and coordination among different types of representations. Understanding an idea in mathematics entails the ability to recognize an idea, which is embedded in a variety of qualitatively different representational systems; the ability to flexibly manipulate the idea within given representational systems and the ability to translate the idea from one system to another accurately (Gagatsis & Shiakalli, 2004).

The results of the study show that proceptual thinkers were able to come up with clear concise solutions of the task unlike procedural thinkers who at times used inappropriate procedures to solve the inequality task. The way in which complicated ideas can become simple to those who conceive them in a focused way is explained beautifully by Fields medallist, William Thurston:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental processes. The insight that goes with this compression is one of the real joys of mathematics.

(Thurston, 1990, p. 847.)

As a consequence, this suggests that teachers need to act as mentors to encourage their students to build thinkable concepts that link together in coherent ways. ‘A’ level mathematics teachers should teach for proceptual understanding. However, if they teach for procedural understanding, there may be long-term drawbacks. The learners may become more procedural and might not develop a process conception, let alone the proceptual conceptions of mathematical concepts. Failure to reify a concept implies that the learners are likely to suffer from cognitive overload leading to eventual failure in further mathematical studies.

In this study Lesson study was used as an evidence-based approach to understand student thinking. The Lesson study, a collaborative approach, which involves the researchers, teachers and students, was used to build pedagogical knowledge and to improve teaching among the ‘A’ level mathematics teachers. Teachers can therefore collaborate among themselves to investigate their own teaching and student learning based on the practice of the lesson study. In the best cases, teachers get important insights into how their students learn from the lesson, where they get stuck, what changes take place, and how they interpret ideas. We believe that observations of student thinking by teachers can provide the kind of data that is directly applicable to making improvements in their lessons.

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