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Spectral Tau Method with Legendre Polynomials to Approximate Second Order Boundary Value Problem

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Abstract

Spectral methods also known as the Method of Weighted Residuals (MWR) are commonly used in many fields such as Mathematics, Engineering, Physics and others. This method is global smooth functions, usually by high order polynomials which differ from the finite element and finite difference which are local smooth functions, usually by low order polynomials. The most popular spectral methods that are commonly used by researchers are Tau, Collocation and Galerkin methods. Since not all the differential equations can be solved analytically, therefore, the numerical solution of the Legendre Tau method is presented. In this study, the Legendre Tau method is proposed and the comparison with the Chebyshev Tau method has been presented. The objectives of this study are to approximate the second order Boundary Value Problem (BVP) using Spectral Tau method by using the Legendre polynomials as the basis function and to make a comparison between the Legendre Tau method with Chebyshev Tau method. The accuracy of the Legendre Tau method is also presented by calculating the absolute error. Besides, the efficiency of both methods was proposed in this study by calculating their CPU times. Previous literature shows that many researchers approximated differential equations using Chebyshev Tau method while the Legendre Tau method has never been used before. The numerical structures established in this study are in line with solutions attained with renowned and standard spectral methods. To validate the results and claim, several test problems were presented in this study.

Keywords: Taumethod, Spectraltaumethod, Legendrepolynomials, Chebyshevpolynomials

Introduction

Many researchers studied various types of numerical techniques for solving the differential equations which are Spectral method, Chebyshev Tau method, Legendre Tau method and other since not all the differential equations cannot be solved analytically. Öztürk (2018) stated that the differential systems and differential equations are suitable tools to discover mathematical equations and mathematical modeling. According to Trefethen (1996) ordinary differential equations (ODE) are easy to solve and understand compared to partial differential equations (PDE) while in the numerical methods for ODE, Runge-Kutta methods and linear multistep methods are the methods that are commonly used.

Furthermore, the trigonometric polynomials such as Chebyshev polynomials, and Legendre polynomials are the trial functions that are recently used in spectral methods (Saldaña et al., 2006). The trial functions were used as the basis functions for a truncated series expansion of the solution. In many areas of mathematics, Chebyshev polynomials are very important especially in approximation theory (Kim, 2012). Khader et al (2015), said that Chebyshev polynomials are the polynomials that are usually used in numerical analysis and mathematical computations.

However, the study conducted by Davari and Ahmadi (2013), which is using the approximation of Legendre polynomials to solve 2nd order linear partial differential equations produced better accuracy. They compared their method with the previous method such as quadratic spline collocation method and Sinc Galerkin method. Next, Jung, Liu et al (2014), also found that in their proposed method using Legendre polynomials is a good approximation to the exact solution. Many researchers employ Chebyshev polynomials and Legendre polynomials in their proposed method. For example, Liu (2009), using Legendre polynomials to develop Adomian decomposition method and the result is compared with Chebyshev polynomials. In addition to that, Hassan et al (2010), proposed ultraspherical tau method employing the series expansions of Chebyshev and Legendre polynomials.

It can be concluded that there are numerous studies conducted about Spectral Tau method employing Legendre polynomials and Chebyshev polynomials. Indeed, many researchers used Chebyshev Tau method to approximate second order BVP as revealed by (Bashir et al., 2015) and (Gourgoulhon et al., 2002). Therefore, the numerical solutions of Legendre Tau method have been proposed in this study since the Legendre Tau method has not been presented by other researchers for approximating the second order Boundary Value Problem (BVP) with Dirichlet boundary conditions. The results then compare with the Chebyshev Tau method with using Chebyshev polynomials as basic functions.

Families of Polynomials

In this subtopic, the properties of Legendre polynomials and Chebyshev polynomials will be presented. These two polynomials are usual families of polynomials that are commonly used by researchers.

Chebyshev Polynomials

The Chebyshev polynomials T_r are an orthogonal set with measure $w = \frac{1}{\sqrt{1-x^2}}$ on [-1, 1]. The scalar product of the two T_r is given by:

$$\int_{-1}^{1} \frac{T_r T_s}{\sqrt{1-x^2}} dx = \frac{\pi}{2}(1+\delta_{0r})\delta_{rs} \tag{3.1}$$

where δ_{rs} is a Kronecker delta same as Legendre polynomials with

$$\delta_{rs} = \begin{cases} 0 \; if \; r \neq s \\ 1 \; if \; r = s \end{cases} \tag{3.2}$$

The polynomials can be figured with $T_0 = 1, T_1 = x$ and the other depends on the recurrence relation which is:

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x), \tag{3.3}$$

Gourgoulhon et al. (2002) claimed that the collocation points are differing to Legendre polynomials since the points of Chebyshev polynomials which can simplify the computational task. Besides, the relation between coefficients a_r of a function u and the coefficients b_r of Lu, where L is a linear operator and it is shown below:
If L is the multiplication by x then:

$$b_r = \frac{1}{2}[(1 + \delta_{0r-1})a_{r-1} + a_{r+1}] \qquad (r \geq 1) \tag{3.4}$$

If L is the derivation:

$$b_r = \frac{2}{(1 + \delta_{0r})} \sum_{\substack{p=r+1 \\ p+r\ odd}}^{N} p a_p \tag{3.5}$$

If L is the second derivation:

$$b_r = \frac{1}{(1 + \delta_{0r})} \sum_{\substack{p=r+1 \\ p+r\ even}}^{N} p(p^2 - r^2)a_p \tag{3.6}$$

Legendre Polynomials
The Legendre polynomials are denoted by Pr establish orthogonal polynomials family on [-1, 1] with w=1 as a measure. One of the advantages of Legendre polynomials is the measure is simple particularly from the analytical approach (Gourgoulhon et al., 2002). The scalar product of the two P_r is given by:

$$\int_{-1}^{1} P_r P_s dx = \frac{2}{2n+1}\delta_{rs} \tag{3.7}$$

where δ_{rs} is a Kronecker delta as given in (3.2)
Given that $P_0 = 1$, $P_1 = x$, and all the other P_r can be obtained by using the recurrence relation as shown below:

$$(r + 1)P_{r+1}(x) = (2r + 1)xP_r(x) - rP_{r-1}(x). \tag{3.8}$$

However, from the formula, it is shown that the collocation points position is not analytical and need to be computed numerically. Some linear operation in coefficient space can be derived. Consider a function u given by its coefficients:

$$u = \sum_{r=0}^{N} a_r P_r(x) \tag{3.9}$$

and H be a linear operator acting on H so that

$$Hu = \sum_{r=0}^{N} b_r P_r(x) \tag{3.10}$$

For some cases, the relation between the a_r and b_r can be written explicitly.
For example:
If L is the multiplication by x:

$$b_r = \frac{r}{2r-1}a_{r-1} + \frac{r+1}{2r+3}a_{r+1} \quad (r \geq 1) \tag{3.11}$$

If L is the derivation:

$$b_r = (2r+1) \sum_{\substack{p=r+1 \\ p+r\ odd}}^{N} a_r \tag{3.12}$$

If L is the second derivation:

$$b_r = (r+\frac{1}{2}) \sum_{\substack{p=r+2 \\ p+r\ even}}^{N} [r(r+1) - r(r+1)]a_r \tag{3.13}$$

Introduction of the Test Problem

The scheme development will apply Legendre polynomials and Chebyshev polynomials as stated from the previous subtopic and will be described more detail in this subtopic. The test problems chosen are second order BVP with Dirichlet boundary conditions with the given exact solution. Since many researchers only used Chebyshev Tau method to approximate second order BVP, therefore, all the three test problems chosen to approximate second order BVP using Legendre Tau method.

Test Problem 1

Consider the following second order BVP (Bashir et al., 2015)

$$u''(x) - 4u'(x) + 4u(x) = e^x - \frac{4e}{1+e^2}, \quad x \in [-1,1] \tag{3.14}$$

with Dirichlet boundary conditions

$$u(-1) = 0, u(1) = 0 \tag{3.15}$$

The boundary value problem has the exact solution which is

$$u(x) = e^x - \frac{\sinh(1)}{\sinh(2)} e^{2x} - \frac{e}{1+e^2} \tag{3.16}$$

Test Problem 2

Consider the following second order BVP (Gheorghiu, 2007)

$$u''(x) + u(x) = x^2 + x, \quad x \in [-1,1] \tag{3.17}$$

with boundary conditions

$$u(-1) = 0, u(1) = 0 \tag{3.18}$$

The boundary value problem has the exact solution which is

$$u(x) = x^2 + x - 2 + \frac{\cos(x)}{\cos(1)} - \frac{\sin(x)}{\sin(1)} \tag{3.19}$$

Test Problem 3

Consider the following second order BVP (Dutykh, 2016)

$$u''(x) + u'(x) - 2u(x) = 2, \quad x \in [-1,1] \tag{3.20}$$

with boundary conditions

$$u(-1) = 0, u(1) = 0 \tag{3.21}$$

The boundary value problem has the exact solution which is

$$u(x) = 1 - \frac{\sinh(2)}{\sinh(3)} e^{x} - \frac{\sinh(1)}{\sinh(3)} e^{-2x} \tag{3.22}$$

The Implementation of Spectral Tau Method for Solving 2nd Order BVP By Using the Basis Function Legendre Polynomials

Consider the following general differential equation:

$$P(x)u'' + Q(x)u' + R(x)u = f(x), \quad x \in [-1, 1] \tag{3.23}$$

$$u(-1) = \alpha, \quad u(1) = \beta \tag{3.24}$$

The linear operator on the l.h.s. of the equation (3.23) can also be written as follows:

$$H = \frac{d^2}{dx^2} + \frac{d}{dx} + Id \tag{3.25}$$

By using the elementary linear operations as (3.25), the matrix representation of H can be constructed that will be useful in the implementation of the different solvers.

Let

$$u = \sum_{i=0}^{N} a_i P_i\,(x) \tag{3.26}$$

Then

$$Hu = \sum_{i=0}^{N} \sum_{j=0}^{N} L_{ij} a_i\, P_i(x) \tag{3.27}$$

The general matrices of derivative operators with respect to the Legendre basis as corresponds to (3.12) and (3.13)

$$\frac{d}{dx} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & \cdots & \cdots \\ 0 & 0 & 3 & 0 & 3 & 0 & & \vdots \\ \vdots & 0 & 0 & 5 & 0 & 5 & & \\ & & & & 7 & \ddots & \ddots & \\ & & & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{(N+1)\times(N+1)} \tag{3.28}$$

$$\frac{d^2}{dx^2} = \begin{pmatrix} 0 & 0 & 3 & 0 & 10 & 0 & 21 & \cdots & \cdots \\ 0 & 0 & 0 & 15 & 0 & 42 & 0 & & \vdots \\ 0 & 0 & 0 & 0 & 35 & 0 & 90 & & \vdots \\ \vdots & & & & & 63 & 0 & \ddots & \vdots \\ \vdots & & & & & & 99 & \ddots & \vdots \\ & & & & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & & & & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & & & & \cdots & 0 \end{pmatrix}_{(N+1)\times(N+1)} \tag{3.29}$$

Therefore, the matrix representation of H was calculated according to (3.25) by substituting the (3.28) and (3.29).

The test functions $\xi_n$are selected to be the same by way of the spectral functions of decomposition. The residual equations by using Legendre polynomial are then

$$(P_r, Hu - f) = 0 \quad \forall r \leq N \tag{3.30}$$

In other words, the equation can be written as by using the matrix H_{ij} which is:

$$\sum_{i=0}^{N} H_{rj} a_j = f \quad \forall n \leq N \tag{3.31}$$

To obtain the value of r.h.s. of test problems, the function $f(x)$ can be expanded as

$$f(x) = \sum_{r=0}^{\infty} a_r P_r(x), \tag{3.32}$$

or in the matrix from

$$[f(x)] = PF \tag{3.33}$$

Then, the Legendre polynomials orthogonality implies that

$$a_r = \frac{2n+1}{2} \int_{-1}^{1} \frac{f(x) P_r(x)}{\sqrt{1-x^2}} dx, \qquad r \geq 1. \tag{3.34}$$

This integral has been computed by using Maple 18 software as shown in Appendix E in order to obtain the values of r.h.s. which is:

$$F = [f_0 \; f_1 \; f_2 \; f_3 \dots f_r]^T \tag{3.35}$$

In the Tau method, the boundary conditions are enforced as extra equations and can be written as:

$$u(x=-1) = 0 \Rightarrow \sum_{j=0}^{N} (-1)^j a_j = 0 \tag{3.36}$$

$$u(x=+1)=0 \Rightarrow \sum_{j=0}^{N} a_j = 0 \tag{3.37}$$

To find an invertible system having a_r as unknown, the last two residual equations are relaxed and substituted by two boundary conditions. The solution comes close to the exact solution if the function is regular. The matrix format of the equation is as follows:

$$\begin{pmatrix} H_{00} & H_{01} & H_{02} & \cdots & \cdots & H_{0N} \\ H_{10} & H_{11} & H_{12} & \cdots & \cdots & H_{1N} \\ \vdots & \vdots & \vdots & \cdots & & \vdots \\ H_{N0} & H_{N1} & H_{N2} & \cdots & \cdots & H_{NN} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_3 \\ \vdots \\ \\ f_r \end{pmatrix} \tag{3.38}$$

The value of $a_0, a_1, a_2 \dots, a_r$ will be obtained by solving the above matrix. Then, the numerical solution by using Legendre Tau method can be obtained as written:

$$u(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + a_3P_3(x) + a_4P_4(x) + \cdots + a_rP_r(x) \tag{3.39}$$

The first few Legendre polynomials are:

$P_0(x) = 1,$
$P_1(x) = x,$
$P_2(x) = \frac{1}{2}(3x^2 - 1),$
$P_3(x) = \frac{1}{2}(5x^3 - 3x),$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$
$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x),$

While the other polynomials can be constructed using the recursion relation as shown in (3.8). Therefore, the approximate solution of $u(x)$ can be written as

$$u_L(x) = a_0 + a_1x + a_2\left(\frac{1}{2}(3x^2 - 1)\right) + a_3\left(\frac{1}{2}(5x^3 - 3x)\right) + \cdots + a_rP_r(x) \tag{3.40}$$

Table 4. 1
The Accuracy of Test Problems between Chebyshev Tau method and Legendre Tau method

	$N = 4$		$N = 8$	
	Chebyshev Tau Method	Legendre Tau method	Chebyshev Tau Method	Legendre Tau method
TEST PROBLEM 1		✓	✓	
TEST PROBLEM 2		✓		✓
TEST PROBLEM 3		✓		✓

Table 4.13 shows the comparison of the accuracy between the numerical solution and absolute error of test problems 1, 2 and 3 on using Chebyshev Tau method and Legendre Tau method with $N = 4$ and $N = 8$. As can be seen from the table, the Legendre Tau method has better accuracy for all the test problems for $N = 4$ compared to the Chebyshev tau method. However, for $N = 8$, the Chebyshev Tau method has better accuracy compared to the Legendre Tau method for test problem 1. Whereas, for test problem 2 and test problem 3, the Legendre Tau method gives more accurate solutions compared to Chebyshev Tau method for $N = 8$. However, it can be concluded that as the number of terms increases, the accuracy of Chebyshev Tau method and Legendre Tau method can be improved since the numerical solution becomes closer to the exact solution and the absolute error gives the smallest value.

Furthermore, the numerical experiments can also be summarized that Legendre Tau method is high efficiency since the CPU times for Chebyshev Tau method are longer for both N for the two test problems. This indicated that Legendre Tau method has better accuracy and efficiency compared to Chebyshev Tau method for certain functions.

Conclusion

The comparison of Chebyshev Tau method and Legendre Tau method has been made in terms of accuracy and efficiency. Since the problems were solved numerically using the spectral Tau method, the numerical results are then compared with the exact solutions. As can be summarized from all the test problems, the Legendre Tau method gives better accuracy rather than Chebyshev Tau method since the numerical solutions of Legendre Tau method are very close to the exact solution for both $N = 4$ and $N = 8$. In addition to that, the absolute error of Legendre Tau method is also smaller than Chebyshev Tau method for both N. It is also observed that the accuracy of the results by using Legendre Tau method can be improved by increasing the number of terms. Besides, Legendre Tau method also shows high efficiency compared to the Chebyshev Tau method since the CPU times for the system to execute is much lesser. This indicates that Legendre Tau method has better accuracy and efficiency compared to the Chebyshev Tau method for certain functions.

The results also show that the solutions express the correct physical behaviour of the model equation between the exact solutions. For further analysis, the solutions still expressed the physical behaviour by using Legendre Tau method same as by Chebyshev Tau method. Therefore, from the analysis, it can be concluded that this method is consistent. Thus, Legendre Tau method is numerically stable as N increases.

This work makes substantial theoretical and contextual contributions to the understanding of spectral methods, especially regarding numerical solutions of differential equations. This study clarifies the benefits of the Legendre approach in terms of accuracy and processing economy by contrasting it with the Chebyshev Tau method. The results are consistent with other studies that emphasize the Legendre Tau method's stability and convergence qualities, which have been demonstrated to produce better approximations in a variety of applications, including parabolic partial differential equations (Saadatmandi & Dehghan, 2010). The study's findings add to the body of knowledge by offering factual proof for the Legendre Tau method's preference in situations requiring a high degree of precision.

Additionally, the study supports the theoretical background of spectral methods by proving that the numerical results are highly dependent on the choice of polynomial basis. This comparative analysis not only supports the results of earlier research (Vyasarayani et al., 2014; Lehotzky & Insperger, 2016) that suggests using Legendre polynomials instead of Chebyshev polynomials in certain situations, but it also creates new opportunities for investigating hybrid approaches that combine the best features of both approaches. Beyond theoretical debates, this work has applications since the Legendre Tau method's proven effectiveness can enhance computational procedures in a variety of scientific and engineering fields. As a result, this study provides a useful manual as well as contributing to the theoretical landscape of numerical approaches.

Recommendation

This study focuses on the comparison between Chebyshev Tau method and Legendre Tau method to solve second order BVP by exploiting the trial functions of Chebyshev polynomials and Legendre polynomials. As mentioned in the previous section, Chebyshev Tau method has been used in numerous research to solve the second order BVP but not for Legendre Tau method because this method is still lacking. So, the recommendation that can be made is the Legendre Tau method used to solve higher-order BVP.

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