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# Fat-tailed Distributions, Value at Risk and the Japanese Stock Market Returns

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**Abstract:** The Japanese economy has been the second largest economy over the world for a long time before the Chinese economy emerged. The Tokyo Stock Exchange (TSE) is the fourth largest stock exchange in the world by aggregate market capitalization of its listed companies and largest in East Asia and Asia. It is of great importance for those in charge of managing risk to understand how its market index returns are distributed. The goal of this paper is to examine how various types of heavy-tailed distribution perform in risk management of the N225 Index returns. We compared these heavy-tailed distributions through a variety of criteria. Our results indicate the generalized hyperbolic distribution has the best goodness of fit and generates most suitable risk measures.
**Keywords:** Financial Risk, Goodness of Fit, Value at Risk

## 1. Introduction

The main elements of Japan's financial system are much the same as those of other major industrialized nations: a commercial banking system, which accepts deposits, extends loans to businesses, and deals in foreign exchange; specialized government-owned financial institutions, which fund various sectors of the domestic economy; securities companies, which provide brokerage services, underwrite corporate and government securities, and deal in securities markets; capital markets, which offer the means to finance public and private debt and to sell residual corporate ownership; and money markets, which offer banks a source of liquidity and provide the Bank of Japan with a tool to implement monetary policy. Although the equity market in Japan is not as important as the equity market in the United States, the Japanese equity market still plays an important role in the Japanese economy.

The Tokyo Stock Exchange (TSE) is the fourth largest stock exchange in the world by aggregate market capitalization of its listed companies and largest in East Asia and Asia. It had 2,292 listed companies with a combined market capitalization of US$4.09 trillion as of April 2015. The Nikkei 225 (N225) is a stock market index for the TSE. It has been calculated daily as a price-weighted index, and the components are reviewed once a year. Currently, the N255 is the most

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widely quoted average of Japanese equities. In the past, severe market distortions have been found in the Japanese equity markets, which highlight the importance of financial risk management practice. For instance, the economy is still suffering somehow from the burst of the bubble end of the 1980s stock trading. In this paper, we focus on quantitative risk management of the equity market in Japan and investigate various types of statistical distributions in capturing tail risk of the equity market. Among these heavy-tailed distributions, we demonstrate that the generalized distribution has superior empirical performance compared several other widely-used heavy-tailed distributions, such as the Student's *t* distribution, the Skewed *t* distribution, and the normal inverse Gaussian distribution (NIG).

**Literature Review**

There are extensive discussions about heavy-tailed distributions on financial risk management as recently reviewed by Ibragimov, Ibragimov, and Walden (2015). In addition to the widely-used Student's *t* distribution, Hansen (1994) introduced a type of Skewed *t* distribution and showed its superior performance in fitting the U.S. Dollar/Swiss Franc exchange rate. Barndorff-Nielsen (1977) introduced generalized hyperbolic distributions into the equity market. Figueroa-Lopez, et al. (2011) investigated estimation of the NIG distribution and demonstrated its applications in high frequency financial data. Finally, Guo (2017a) compared the above mentioned heavy-tailed distribution, and showed the Skewed *t* distribution provides the best goodness-of-fit and risk management measures for the stock market in the United States.

Following Guo (2017a, 2017b), we re-investigate the topic of heavy-tailed distributions in financial risk management, but focus on the equity market in Japan. There are extensive researches on the topic of heavy tails of the Japanese stock market returns. For instance, Gettinby, et al. (2006) seeked to characterize the distribution of extreme returns for US, UK and Japanese equity indices over the years 1963–2000, and found that the generalized logistic distribution has the best fitting among the distributions studied for all three countries over the period of study. Jondeau and Rockinger (2003) studied similarities between the left and right of daily stock-market returns for 20 countries, including Japan, and verified that the perception that left tails are heavier than right ones is not due to clustering of extremes, but likely due to the relative infrequency of large extremes. There are some other researches investigating the relationship between the Japanese stock market and other stock markets. Poon, Rockinger and Tawn (2003) studied cross-sectional dependence in extreme returns of the stock market indexes among the US, UK, Germany, France and Japan, and found that tail dependence among these stock indexes can be partially captured by models with heteroskedasticity. Azad (2009) investigated whether East Asian stock markets, China, Japan and South Korea, are individually and/or jointly efficient, and whether contagion exists between these markets. Azad found that compared with the Chinese stock market, the Japanese and South Korean stock markets are more isolated.

In this paper, we compare the various types of heavy-tailed distribution using the Japanese stock market index returns. We focus on the quantitative risk management practices that examine which statistical distribution generates most suitable stress test scenarios. The remainder of the paper is organized as follows. In Section 2, we introduce the heavy-tailed distributions. Section 3 discusses the data. The estimation results are in Section 4. Finally, we conclude in Section 5.

## 2. The Heavy-tailed Distributions

Following Guo (2017), we are interested the four following types of heavy-tailed distribution: (i) the Student's *t* distribution; (ii) the Skewed *t* distribution; (iii) the normal reciprocal inverse Gaussian distribution (NIG); and (iv) the generalized hyperbolic distribution (GH). All the distributions have been standardized to ensure mean and standard deviation equal to zero and one respectively. We also consider the Gaussian distribution as a benchmark distribution. Their probability density functions are given as follows.

(i) Student's *t* distribution:

$$f(e_t \mid \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})[(\nu-2)\pi]^{1/2}} \left(1 + \frac{e_t^{\,2}}{(\nu-2)}\right)^{-\frac{\nu+1}{2}}, \tag{1}$$

where $\nu$ indicates degrees of freedom and $e_t$ is daily US equity market index return.

(ii) The Skewed *t* distribution is specified as in Hansen (1994). The Skewed *t* distribution:

$$f(e_t \mid \nu, \beta) = \begin{cases} bc\left(1 + \frac{1}{\nu-2}\left(\frac{be_t + a}{1-\beta}\right)^2\right)^{-(\nu+1)/2} & e_t < -a/b \\ bc\left(1 + \frac{1}{\nu-2}\left(\frac{be_t + a}{1+\beta}\right)^2\right)^{-(\nu+1)/2} & e_t \geq -a/b \end{cases}, \tag{2}$$

where $e_t$ is the standardized log return, and the constants $a$, $b$ and $c$ are given by $a = 4\beta c\left(\frac{\nu-2}{\nu-1}\right)$, $b^2 = 1 + 3\beta^2 - a^2$, and $c = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi(\nu-2)}\Gamma(\frac{\nu}{2})}$. The density function has a mode of $-a/b$, a mean of zero, and a unit variance. The density function is skewed to the right when $\beta > 0$, and vice-versa when $\beta < 0$. The Skewed *t* distribution specializes to the standard Student's *t* distribution by setting the parameter $\beta = 0$.

(iii) Normal inverse Gaussian distribution (NIG):

$$f(e_t \mid \mu, \alpha, \beta, \delta) = \frac{\alpha\delta K_1(\alpha\sqrt{\delta^2 + (e_t - \mu)^2})}{\pi\sqrt{\delta^2 + (e_t - \mu)^2}} \exp(\delta\sqrt{\alpha^2 - \beta^2} + \beta(e_t - \mu)), \tag{3}$$

where $K_{\lambda}(\cdot)$ is the modified Bessel function of the third kind and index $\lambda = 0$ and $\alpha > 0$. The NIG distribution is specified as in Prause (1997). The NIG distribution is normalized by setting $\mu = -\frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}$ and $\delta = \frac{\left(\sqrt{\alpha^2 - \beta^2}\right)^3}{\alpha^2}$, which implies $E(e_t) = 0$ and $Var(e_t) = 1$.

(iv) Generalized hyperbolic distribution:

$$f\left(e_t \mid p, b, g\right) = \frac{g^p}{\sqrt{2\pi}\left(b^2 + g^2\right)^{\frac{1}{2}\left(p-\frac{1}{2}\right)} d\left(p,b,g\right) K_p\left(g\right)} q\left(\frac{e_t - m\left(p,b,g\right)}{d\left(p,b,g\right)};\ p,b,g\right), \qquad (4)$$

where $\tilde{R}_n \equiv \frac{K_{n+p}\left(g\right)}{g^n K_p\left(g\right)}$, $d\left(p,b,g\right) \equiv \left[\tilde{R}_1 + b^2\left\{\tilde{R}_2 - \tilde{R}_1\right\}\right]^{-\frac{1}{2}} \geq 0$, and $m\left(p,b,g\right) \equiv -b\, d\left(p,b,g\right)\tilde{R}_1$. $p, b$ and $g$ are parameters. The generalized hyperbolic distribution is a standardized version of Prause (1997).

## 3. Data

We fit the heavy tailed distributions with the normalized Japanese equity market index returns. The N225 is a price-weighted stock market index for the Tokyo Stock Exchange that includes the top 225 blue-chip companies listed on the Tokyo Stock Exchange. Some of the companies listed on the Asia's oldest index are Canon Inc., Panasonic Corp., Sony Corp., Nissan Motor Co. or Toyota Motor Corp. Companies in the technology sector account for over 40%. We collected the standardized N225 daily dividend-adjusted close returns from Yahoo Finance for the period from January 5, 1984 to July 18, 2017, covering all the available data in Yahoo Finance. There are in total 8245 observations. Figure 1 illustrates the dynamics of the N225 returns, and the figure exhibits significant volatility clustering.

**Figure 1**: N225 returns

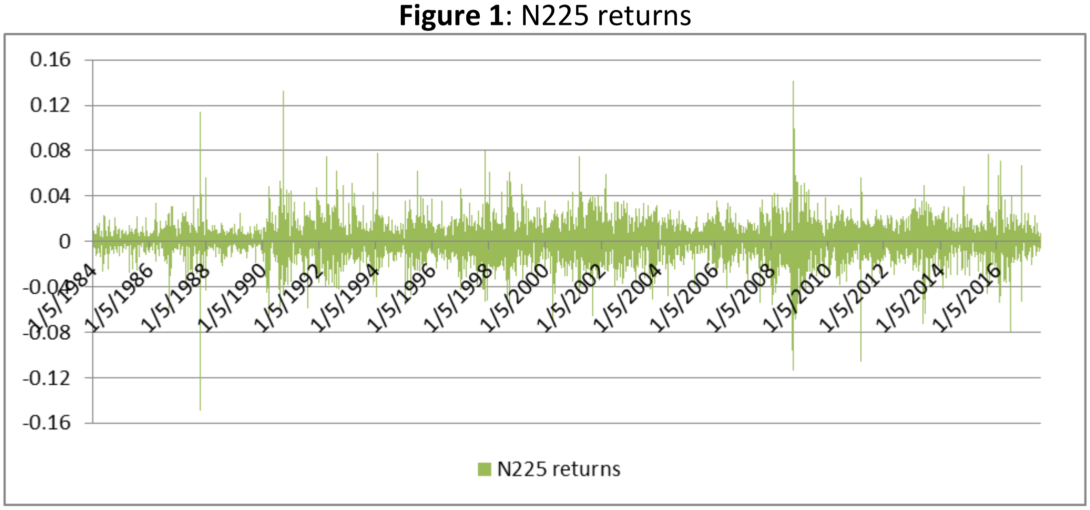


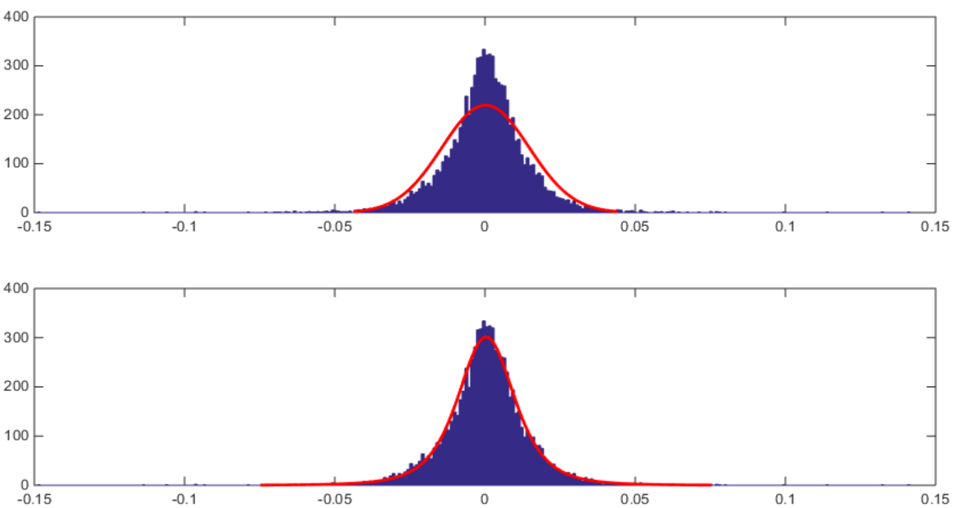

Table 1 presents basic statistics of the N225 returns. The results show the N225 daily returns are leptokurtotic and negatively skewed. The extreme downside move is almost twice of the extreme upside move.

**Table 1:** Descriptive statistics

| min | max | mean | std | skewness | kurtosis |
|---|---|---|---|---|---|
| -14.90% | 14.15% | 0.02% | 1.45% | -0.07 | 7.65 |

Figure 2 is the histogram of the raw data. We fit the returns by the Gaussian distribution and the Student's *t* distribution. The upper panel in Figure 2 is fitted by the normal distribution. Clearly, the Student's *t* distribution has a much better in-sample goodness of fit.

**Figure 2:** N225 returns - Normal vs. Student's *t*




## 4. Empirical Results

### 4.1 Parameters Estimation

We estimated the parameters by the maximum likelihood estimation (MLE) method and the estimation results of the key parameters are given in Table 2. All the parameters are significantly different from zero at 10% significance level.

**Table 2:** Estimated values of key parameters

| | Normal | Student's t | Skewed t | NIG | Generalized Hyperbolic |
|---|---|---|---|---|---|
| Symmetric | Y | Y | N | N | N |
| Fat-tailed | N | Y | Y | Y | Y |
| Estimated Parameters | | Nu=5.03 | Nu=5.06; beta=0.002 | alpha=1.63; beta=-0.025 | p=-1.367; b=-.012; g=0.403 |

### 4.2 Goodness of Fit

Following Huber-Carol, et al. (2002) and Taeger and Kuhnt (2014), in this section we compare the four heavy-tailed distributions and the benchmark normal distribution in fitting the N225 daily returns through four different criteria: (i) Kolmogorov-Smirnov statistic; (ii) Cramer-von Mises criterion; (iii) Anderson-Darling test; and (iv) Akaike information criterion (AIC).

(i) Kolmogorov-Smirnov statistic is defined as the maximum deviation between empirical CDF (cumulative distribution function) $F_n(x)$ and tested CDF $F(x)$:

$$D_n = \sup_x |F_n(x) - F(x)|, \tag{5}$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{[-\infty,x]}(X_i)$.

(b) Cramer-von Mises criterion is defined as the average squared deviation between empirical CDF and tested CDF:

$$T = n\int_{-\infty}^{\infty}[F_n(x) - F(x)]^2 dF(x) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{2i-1}{2n} - F_n(x_i)\right]^2, \tag{6}$$

(c) Anderson-Darling test is defined as the weighted-average squared deviation between empirical CDF and tested CDF:

$$A = n\int_{-\infty}^{\infty}\frac{(F_n(x) - F(x))^2}{F(x)(1-F(x))}dF(x),$$

and the formula for the test statistic $A$ to assess if data comes from a tested distribution is given by:

$$A^2 = -n - \sum_{i=1}^{n}\frac{2i-1}{n}\left[\ln(F(x_i)) + \ln(1-F(x_i))\right]. \tag{7}$$

(d) Akaike information criterion (AIC) is defined as:

$$AIC = -2k - 2\ln(L), \tag{8}$$

where $L$ is the maximum value of the likelihood function for the model, and $k$ is the number of estimated parameters in the model.

The comparison results are showed in Table 3, indicating the generalized hyperbolic distribution has the best goodness of fit compared with other selected types of distribution, followed by the Student's *t* distribution and the Skewed *t* distribution.

| | Normal | Student's *t* | Skewed *t* | NIG | Generalized Hyperbolic |
|---|---|---|---|---|---|
| K-S Test | 0.013 | 0.007 | 0.007 | 0.008 | 0.006 |
| Cv-M Test | 0.027 | 0.022 | 0.021 | 0.022 | 0.019 |
| A-D Test | 1.37 | 1.13 | 1.06 | 1.10 | 1.04 |
| AIC | 26504 | 25314 | 25134 | 25546 | 24876 |

**Table 3**: Comparison of selected types of Distribution

### 4.3 Value at Risk

In financial mathematics and financial risk management, VaR is defined as: for a given position, time horizon, and probability *p*, the *p* VaR is defined as a threshold loss value, such that the probability that the loss on the position over the given time horizon exceeds this value is *p*. with the estimated parameters in Section 4.1, we calculate VaRs for different confidence levels:

$$VaR_{\alpha}(e_t) = \inf\{e \in \mathbb{R} : P(e_t > e) \le 1-\alpha\}, \quad (9)$$

where $\alpha \in (0,1)$ is the confidence level. We select the following levels for downside moves: {99.999%, 99.99%, 99.97.5%, 99.95%, 99.9%}, and for upside moves: {0.001%, 0.01%, 0.025%, 0.05%, 0.1%}. From Equation (9), the VaR levels are given as in Table 4. Table 4 indicates that the generalized hyperbolic distribution has the closest VaRs to the nonparametric historical VaRs compared with other types of distributions.

**Table 4**: Scenarios for N225 shocks

| Left Tail | | | | | |
|---|---|---|---|---|---|
| Confidence | 99.99% | 99.95% | 99.90% | 99.50% | 99.00% |
| Empirical | -13.59% | -12.42% | -9.89% | -7.67% | -6.34% |
| Normal | -7.94% | -7.07% | -6.62% | -5.89% | -5.44% |
| T | -13.02% | -11.67% | -10.67% | -8.74% | -7.39% |
| Skewed T | -13.07% | -11.87% | -10.19% | -8.22% | -6.82% |
| NIG | -11.57% | -10.72% | -9.34% | -8.57% | -7.32% |
| GH | -13.44% | -12.49% | -9.52% | -7.82% | -6.57% |
| Right Tail | | | | | |
| Confidence | 0.01% | 0.05% | 0.10% | 0.50% | 1.00% |
| Empirical | 13.52% | 11.27% | 10.47% | 8.77% | 7.77% |
| Normal | 7.94% | 7.07% | 6.62% | 5.89% | 5.44% |
| T | 13.02% | 11.67% | 10.67% | 8.74% | 7.39% |
| Skewed T | 11.94% | 9.62% | 8.77% | 6.92% | 5.92% |
| NIG | 12.59% | 11.27% | 9.59% | 8.42% | 7.32% |
| GH | 13.47% | 11.49% | 9.89% | 8.32% | 6.69% |

## 5. Conclusions

The trade-off between risk and return is often taken for granted. However, if market risk could be appropriately managed, market participants do not need suffer from serious losses. In this paper, we are particularly interested in the Japanese stock market. We first show the Japanese stock index returns exhibit significant fat tails and marginal negative skewness. Then, taking advantage of the results in Guo (2017a) and Guo (2017c), the market return series is fitted through several widely-used heavy-tailed distributions: (i) the Student's *t* distribution; (ii) the Skewed *t* distribution; (iii) the normal reciprocal inverse Gaussian distribution (NIG); and (iv) the generalized hyperbolic distribution (GH). Our results indicate the generalized hyperbolic distribution has the best goodness of fit and generates most suitable risk measures.

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