Black-Scholes Models with Inherited Time and Price Memory

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Abstract

In this study, we examine three Black-Scholes option pricing models with inherited time and assess price memories. In other words, we take into consideration the reliance of these models on both the current state and the history of changes. A solution in the form of a bivariate time-price fractional power series is proposed for these models to study the shared influence of the memory index in both the time and price values. The findings of this study agree with the existing solutions in the literature for the classical option pricing integer-order cases. Therefore, the suggested scheme is strongly recommended to study further memory indexed financial models.

Keywords

Fractional Black-Scholes, Memory index, Fractional time-price power series solution

1. Introduction

Option valuation plays a vital role on derivative markets and financial investment. Under certain hypotheses, Black and Scholes derived a theoretical valuation model for options that relies only on the price of the stock, time, and variables that are considered to be known scalars. Therefore, in option and stock ideal scenario, "it is possible to create a hedged position, consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock, but will depend only on time and the values of known constants" (Black and Scholes, 1973). This model remains the hallmark of option trading and provides a mathematical validity to the activities of the European and American call or put options.

In fact, the Black-Scholes model is financial stochastic differential equations in nature. However, based on a combination of dynamic hedging technique, Ito calculus, and the no-arbitrage principle, it is demonstrated that these models can be transformed to linear evolutionary partial differential equations with variable coefficients (Gazizov and Ibragimov, 1998), somewhat similar to the well-known diffusion equation, of the form

$$\frac{\partial W(x,t)}{\partial t} + \frac{\xi^2(x,t)}{2} x^2 \frac{\partial^2 W(x,t)}{\partial x^2} + r(t)\, x\, \frac{\partial W(x,t)}{\partial x} - r(t)\, W(x,t) \,=\, 0 \tag{1.1}$$

Where: $W(x,t)$ the European call option price at asset price $x \in \mathbb{R}_+$ and at time $t \in (0,T)$, T is the maturity, $r(t)$ is the risk free interest rate, and $\xi(x,t)$ is the volatility function of underlying asset. If E represents the expiration price for the option, then the value of the European call option $W_c(x,t) = \max\{x - E, 0\}$ and the European put option $W_p(x,t) = \max\{E - x, 0\}$.

Recently, with the revolution of fractional calculus, it has been demonstrated that the non-integer order derivatives can reflect memory and heredity features ingrained in the processes (Du *et al.*, 2013). Moreover, this memory phenomenon is not necessarily accompanied to the time variable, but also it could be accompanied by other variables (Pandey *et al.*, 2016). Our motivation in this work is to look at the Black-Scholes model in a wider frame by replacing all the integer derivatives in equation (1.1) by non-integer ones. In other words, we aim to examine analytic solutions of the Black-Scholes model with inherited memory in both the time and price coordinates.

The rest of the paper is organized as follows: In Section 2, we propose a solution representation for Black-Scholes option pricing models with inherited time and assess price memories. In Section 3, we employ the corresponding power series method to extract closed-form solutions for three Black-Scholes option pricing models. Finally, some concluding remarks are provided in Section 4.

2. A solution representation of Black-Scholes models

In this section, we provide a solution representation for Black-Scholes option pricing models with inherited time and assess price memories in terms of a bivariate fractional power series.

Definition 2.1. Let $n - 1 < \alpha < n \in \mathbb{N}$ and $W(x,t)$ be an appropriate time-price function. Then the time–fractional derivative, in Caputo sense, of order $\alpha > 0$ is defined by

$$\frac{\partial^\alpha W(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n w(x,\tau)}{\partial \tau^n} \, d\tau. \tag{2.1}$$

Similarly, we can define the price–fractional derivative of $W(x,t)$. A direct consequence of the last definition with $n - 1 < \alpha < n$ yields that

$$\frac{\partial^\alpha t^a}{\partial t^\alpha} = \begin{cases} \dfrac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} t^{a-\alpha}, & a > n-1, \, a \in \mathbb{R} \\ 0 & , \quad a \leq n-1, \, a \in \mathbb{N} \end{cases} \tag{2.2}$$

Further, it enough to restrict our attention to the case where $\alpha \in (0,1)$ since

$$\frac{\partial^\alpha W(x,t)}{\partial t^\alpha} = \frac{\partial^{\alpha-(n-1)} W^{(n-1)}(x,t)}{\partial t^{\alpha-(n-1)}}$$

for an arbitrary fractional derivative order $n - 1 < \alpha < n$, where $\alpha - (n - 1) \in (0,1)$.

Definition 2.2. A time-price fractional power series representation centered at origin is an infinite series of the form

$$\underbrace{\sum_{i+k=0}^{\infty} \xi_{ik}\, t^{i\alpha} x^{k\gamma}}_{} = \underbrace{\xi_{00}}_{i+k=0} + \underbrace{\xi_{10}\, t^\alpha + \xi_{01}\, x^\gamma}_{i+k=1} + \underbrace{\xi_{20}\, t^{2\alpha} + \xi_{11}\, t^\alpha x^\gamma + \xi_{02}\, x^{2\gamma}}_{i+k=2} + \cdots, \tag{2.3}$$

where $i,k \in \mathbb{N}^*$, $t,x \geq 0$ are variables, and $\xi_{ik} \in \mathbb{R}$ are coefficients of the series.

Theorem 2.3. *(Jaradat et al., 2018) Suppose that the time-price function $W(x,t)$ can be written in the form* (2.3) *on* $\Lambda = \{(t,x) : 0 \leq t < R_t, 0 \leq x < R_x\}$. *If* $\frac{\partial^{n\alpha}}{\partial t^{n\alpha}} \frac{\partial^{m\gamma}}{\partial x^{m\gamma}} [W(x,t)] \in \mathcal{C}((0,R_t) \times (0,R_x))$ *for n,m* $\in \mathbb{N}$, *then for all $(t,x) \in \Lambda$, α, $\gamma \in (0,1)$, and $r, s \in \mathbb{N}$ we have*

$$\frac{\partial^{n\alpha}}{\partial t^{n\alpha}} \frac{\partial^{m\gamma}}{\partial x^{m\gamma}} [W(x,t)] = \sum_{\substack{i+k=n+m \\ i \geq n, \, k \geq m}}^{\infty} \xi_{ik} \frac{\Gamma(i\alpha+1)\Gamma(k\gamma+1)}{\Gamma((i-n)\alpha+1)\Gamma((k-m)\gamma+1)} t^{(i-n)\alpha} x^{(k-m)\gamma}$$

$$= \sum_{i+k=0}^{\infty} \xi_{i+n,k+m} \frac{\Gamma((i+n)\alpha+1)\Gamma((k+m)\gamma+1)}{\Gamma(i\alpha+1)\Gamma(k\gamma+1)} t^{i\alpha} x^{k\gamma}. \tag{2.4}$$

Consequently, by substituting $x = 0$ and $t = 0$ into (2.4), we obtain the following form for the coefficients

$$\xi_{ik} = \frac{\frac{\partial^{i\alpha}}{\partial t^{i\alpha}} \frac{\partial^{k\gamma}}{\partial x^{k\gamma}} \left[W(0,0) \right]}{\Gamma(i\alpha + 1)\Gamma(k\gamma + 1)}. \tag{2.5}$$

3. An analytical solution of Black-Scholes models

In this section, we examine three Black-Scholes option pricing models with inherited time and price memories and provide their solutions analytically in terms of a fractional power series representation that endowed with time and price fractional derivative orders. Ultimately, the solutions are given in closed-forms that agree with the existing solutions in the literature for the integer-order cases. In all our models, we assume that $t, x \geq 0$.

Model 1. Consider the Black-Scholes model with time and price memory indices $\alpha, \gamma \in (0,1)$:

$$\frac{\partial^\alpha W(x,t)}{\partial t^\alpha} + x^{2\gamma}\frac{\partial^{2\gamma}W(x,t)}{\partial x^{2\gamma}} + 0.5x^\gamma\frac{\partial^\gamma W(x,t)}{\partial x^\gamma} - W(x,t) = 0 \tag{3.1}$$

subject to the initial condition

$W(x,0) = x^{3\gamma}. \tag{3.2}$

We seek a solution to (3.1) and (3.2) in the expansion form

$$W(x,t) = \sum_{i+k=0}^{\infty} \xi_{ik}\, t^{i\alpha}\, x^{k\gamma}. \tag{3.3}$$

By implementing the initial condition into (3.3), we acquire the initial solution terms $\xi_{03} = 1$ and $\xi_{0k} = 0$ for $k \in N^0 - \{3\}$. Now, plugging all the related formulas (2.4) into (3.1) and balancing the analogous terms to get the following difference formulas for ξ_{ik}:

$$\frac{\Gamma((i+1)\alpha + 1)}{\Gamma(i\alpha + 1)}\xi_{i+1,k} - \xi_{ik} = 0; k = 0$$

$$\frac{\Gamma((i+1)\alpha + 1)}{\Gamma(i\alpha + 1)}\xi_{i+1,k} + \left(0.5\,\Gamma(\gamma + 1) - 1\right)\xi_{ik} = 0; \quad k = 1 \tag{3.4}$$

$$\frac{\Gamma((i+1)\alpha + 1)}{\Gamma(i\alpha + 1)}\xi_{i+1,k} + \left(\frac{\Gamma(k\gamma + 1)}{\Gamma((k-2)\gamma + 1)} + \frac{0.5\,\Gamma(k\gamma + 1)}{\Gamma((k-1)\gamma + 1)} - 1\right)\xi_{ik} = 0; \quad k \geq 2$$

Solving recursively equations (3.4) for the coefficients ξ_{ik}, we acquire

$$\xi_{i3} = \left(1 - \frac{\Gamma(3\gamma + 1)}{\Gamma(\gamma + 1)} - \frac{0.5\,\Gamma(3\gamma + 1)}{\Gamma(2\gamma + 1)}\right)^i \frac{1}{\Gamma(i\alpha + 1)}; \quad i \geq 0 \tag{3.5}$$

$\xi_{ik} = 0$; otherwise.

Thus, the time-price memory solution of Black-Scholes model (3.1) – (3.2) is given in a closed-form as

$$W(x,t) = \sum_{i=0}^{\infty} \xi_{i3}\, t^{i\alpha} x^{3\gamma}$$

$$= x^{3\gamma}\sum_{i=0}^{\infty}\left(1 - \frac{\Gamma(3\gamma + 1)}{\Gamma(\gamma + 1)} - \frac{0.5\,\Gamma(3\gamma + 1)}{\Gamma(2\gamma + 1)}\right)^i \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}$$

$$= x^{3\gamma} E_\alpha\left(\left(1 - \frac{\Gamma(3\gamma + 1)}{\Gamma(\gamma + 1)} - \frac{0.5\,\Gamma(3\gamma + 1)}{\Gamma(2\gamma + 1)}\right)t^\alpha\right), \tag{3.6}$$

Where: $E_\alpha(\cdot)$ is the well-known Mittag-Leffler function. We remark here as $\gamma \to 1$, we acquire the time-fractional solution of (3.1)–(3.2)

$W(x,t) = x^3 E_\alpha(-6.5t^\alpha), \tag{3.7}$

Which is in agreement with the solution provided in (Edeki *et al.*, 2017) using the projected differential transformation method. Moreover, as $\alpha, \gamma \to 1$, we acquire the solution for the Black-Scholes integer-order model of (3.1)–(3.2)

$$W(x,t) = x^3 \sum_{i=0}^{\infty} \frac{(-6.5t)^i}{i!}$$
$$= x^3 e^{-6.5t}. \tag{3.8}$$

Model 2. Consider the Black-Scholes model with time and price memory indices $\alpha, \gamma \in (0,1)$:

$$\frac{\partial^\alpha W(x,t)}{\partial t^\alpha} = \frac{\partial^{2\gamma} W(x,t)}{\partial x^{2\gamma}} + (\lambda - 1)\frac{\partial^\gamma W(x,t)}{\partial x^\gamma} - \lambda W(x,t) \tag{3.9}$$

subject to the initial condition
$W(x,0) = E_\gamma(x^\gamma) - 1.$ (3.10)
Again, we assume the solution of (3.9) and (3.10) has the form

$$W(x,t) = \sum_{i+k=0}^{\infty} \xi_{ik} t^{i\alpha} x^{k\gamma}. \tag{3.11}$$

By implementing the initial condition into (3.11), we acquire the initial solution terms $\xi_{00} = 0$ and $\xi_{0k} = \frac{1}{\Gamma(k\gamma+1)}$ for $k \geq 1$. Now, plugging all the related formulas (2.4) into (3.9) and balancing the analogous terms to get the following difference formula for the coefficients ξ_{ik}:

$$\frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)}\xi_{i+1,k} = \frac{\Gamma((k+2)\gamma+1)}{\Gamma(k\gamma+1)}\xi_{i,k+2} + (\lambda-1)\frac{\Gamma((k+1)\gamma+1)}{\Gamma(k\gamma+1)}\xi_{i,k+1} - \lambda\xi_{ik}. \tag{3.12}$$

Solving recursively equations (3.12) for the coefficients ξ_{ik}, we acquire

$$\xi_{i0} = \frac{(-1)^{i+1}\lambda^i}{\Gamma(i\alpha+1)}; \quad i \geq 0$$
$$\xi_{0k} = \frac{1}{\Gamma(k\gamma+1)}; \quad k \geq 1 \tag{3.13}$$
$\xi_{ik} = 0; \quad$ otherwise.

Thus, the time-price memory solution of Black-Scholes (3.9)–(3.10) is given in a closed-form as

$$W(x,t) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}\lambda^i}{\Gamma(i\alpha+1)} t^{i\alpha} + \sum_{k=1}^{\infty} \frac{1}{\Gamma(k\gamma+1)} x^{k\gamma}$$
$$= E_\gamma(x^\gamma) - E_\alpha(\lambda t^\alpha). \tag{3.14}$$

We remark here as $\gamma \to 1$, we acquire the time-fractional solution of (3.9)-(3.10)

$W(x,t) = e^x - E_\alpha(\lambda t^\alpha),$ (3.15)

Which is in agreement with the solution provided in (Elbeleze *et al.,* 2013) using a combination of the homotopy perturbation method, Sumudu transform, and He's polynomials. Moreover, as $\alpha, \gamma \to 1$, we acquire the solution for the Black-Scholes integer-order model of (3.9)–(3.10)

$W(x,t) = e^x - e^{\lambda t},$ (3.16)

Which is in agreement with the solution provided in (Edeki *et al.,* 2015) using the projected differential transformation method.

Model 3. Finally, we consider the following Black-Scholes model with time memory index $\alpha \in (0,1)$:

$$\frac{\partial^\alpha W(x,t)}{\partial t^\alpha} + 0.08(2 + \sin(x))^2 x^2 \frac{\partial^2 W(x,t)}{\partial x^2} + 0.06x\frac{\partial W(x,t)}{\partial x} - 0.06W(x,t) = 0 \tag{3.17}$$

Subject to the initial condition
$$W(x,0) = \max\{x - 25e^{-0.06}, 0\}. \tag{3.18}$$

Since the model endowed with only time memory index α, we can assume that the solution to (3.17)–(3.18) has the form

$$W(x,t) = \sum_{i=0}^{\infty} \xi_i(x)\, t^{i\alpha}.$$ (3.19)

Therefore,

$$\frac{\partial^n W(x,t)}{\partial x^n} = \sum_{i=0}^{\infty} \xi_i^{(n)}(x)\, t^{i\alpha}$$ (3.20)

and by using the fact (2.2), we have

$$\frac{\partial^\alpha W(x,t)}{\partial t^\alpha} = \sum_{i=0}^{\infty} \xi_{i+1}(x) \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)}\, t^{i\alpha}.$$ (3.21)

By implementing the initial condition into (3.19), we acquire the initial solution terms $\xi_0(x) = \max\{x - 25e^{-0.06}, 0\}$. Now, plugging (3.19), (3.20), and (3.21) into (3.17) and balancing the analogous terms to get the following difference-differential equation for the coefficients $\xi_i(x)$:

$$\frac{\Gamma\big((i+1)\alpha+1\big)}{\Gamma\big(i\alpha+1\big)} \xi_{i+1}(x) + 0.08\big(2+\sin(x)\big)^2 x^2\, \xi_i''(x) + 0.06x\, \xi_i'(x) - 0.06\xi_i(x) = 0.$$ (3.22)

In recursive manner, this consequently implies that for $i \geq 1$

$$\xi_i(x) = \Big(\max\{x - 25e^{-0.06}, 0\} - x \Big) \frac{(0.06)^i}{\Gamma\big(i\alpha+1\big)}.$$ (3.23)

Thus, the time memory solution of the Black-Scholes (3.17)–(3.18) is given in a closed-form as

$$W(x,t) = \max\{x - 25e^{-0.06}, 0\} + \Big(\max\{x - 25e^{-0.06}, 0\} - x \Big) \sum_{i=1}^{\infty} \frac{(0.06)^i}{\Gamma\big(i\alpha+1\big)}\, t^{i\alpha}$$

$$= x + \Big(\max\{x - 25e^{-0.06}, 0\} - x \Big) E_\alpha\big(0.06t^\alpha\big)$$ (3.24)

Which is in agreement with the solution provided in (Elbeleze, Kilicman, &Taib, 2013). In particular, if $\alpha \to 1$, we have the exact solution

$$W(x,t) = x + \Big(\max\{x - 25e^{-0.06}, 0\} - x \Big) e^{0.06t}$$ (3.25)

to the classical Black-Scholes of (3.17)–(3.18) (Edeki *et al.*, 2015).

4. Conclusions

In this study, we have provided analytical solutions for three Black-Scholes option pricing models endowed with memory index on time and price simultaneously. To the author's knowledge, this is the first attempt to examine the mutual impact of the time and price fractional derivatives of BlackScholes models. The solutions are given in the form of convergent fractional power series with easily accessible coefficients and without any restrictive presumptions. Moreover, these solutions generalized all the existing results regarding the integer and time-fractional versions of Black-Scholes models. This gives the opportunity to predict the European call option price from the time and assess price changes in the past.

Contribution

The outcomes of this research submit various contributions. For the theoretical side, the analytical scheme conducted in this study has shown the generality and independence of the Black-Scholes option pricing models with multi-memory indices. To the best knowledge of the author, this is the first attempt to consider Black-Scholes models in the proposed format where the time and the price endowed with fractional parameters. Further, analytical solutions have successfully obtained in terms of convergence power series in a fractional sense. From the side of practitioners, the obtained solutions of these models would contribute to utilizing the history of the European call option price to predict the present valuation for options.

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